# 1. The primal problem

**Sets**Number

**Parameters**

**Variables**

We will use i for the companies, and j for the queries, with i=1,2,……,n and j=1,2,…..m. We will call Cij to the revenue per user ad for each query j from company i and the variables will be denoted as Xij, which refer to the total number of ads from company i for each query j.

Moreover we will use bi for the maximum budget of company i, with i=1,2,….,n y dj for the maximum estimated number of request of query j, para j=1,2,…..,m.

Using this notation, the general formulation of this problem will be the following:

s.t

(1)

(2)

# 2. The standard formulation

We are going to obtain the equivalent standard form. The objective function would be:

In this case i=1,2,…,n and j=1,2,…,m and also:

The objective function that we have to maximize would be:

For the condition (1), it is necessary to construct the following matrix:

In this way, we can express the restrictions given in the expression (1) in matrix format:

So as to obtain the matrix form of the conditions given in (2), the following matrix is used:

m columns

m columns

m columns

NOTE: Each block of the previous matrix has m columns.

With this matrix, we successfully satisfy the conditions given in (2), that is:

By joining the two previous expressions and using a matrix block notation, the next general expression will be obtained.

We need to add the Z variables to transform the inequality constraints into equalities, obtaining in this way the standard formulation of the problem. This will be done in the following sections, where the formulation of the dual problem, generalizing this formulation for a model of any size.

Before continuing, it should be noted that in this type of problems, we have n restrictions derived from the maximum budget of each company and m restrictions derived from the estimated number of times each query will be requested over a given day. Then, we are going to add positive variables, that we will call them Z1i (i=1,2,...,n), to the first n restrictions, in order to transform the inequality into an equality. And we add the positive variables Z2j (j = 1, ..., m) to the other m constraints also to convert these constraints into equalities.

In this situation, we add two different positive variables, that are called slack variables for this kind of problems. On the one hand, we have Z1i (i=1,2,...n), Which are the ones that transform the restrictions concerning the maximum budget of each enterprise into equalities. And on the other hand, we have the Z2j (j=1,2,...,m), which are the ones that also transform the restrictions regarding the estimated number of queries into equalities.

With this approach, will have the following information:

X1=(X11,....,X1,m;X21,......,X2,m;........,Xn,1,.......,Xn,m,Z11,....Z1n,Z21,.....,Z2m)t

n terms

m terms

CT=(-C11,...,-C1,m;-C21,...,-C2,m;...;-Cn,1,....,-Cn,m, 0, .... , 0, 0, ....., 0)

The matrices A1 y A2 will have the following structure:

m columns

m columns

m columns

m columns

m columns

The same way as before, we construct the matrix A using block matrix form:

And we call b1 to the vector that contains all the restrictions. Using again the block matrix form, the vector b1 is represented as follows:

Therefore, taking everything into account, the primal problem will be:

s.t. AX1=b1

# 3. Results and sensitivity analysis

The input data used to solve this problem is given in the following table:

|  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Queries | | | | | | | | | | Budget |
| Companies | Q1 | Q2 | Q3 | Q4 | Q5 | Q6 | Q7 | Q8 | Q9 | Q10 |
| C1 | 1 | 0.75 | 2.5 | 2 | 3 | 3.5 | 1.75 | 0.5 | 2.5 | 1.75 | 200 |
| C2 | 0.5 | 0.5 | 2 | 0.75 | 1.5 | 1 | 0.6 | 0.9 | 1.5 | 2 | 150 |
| C3 | 0.5 | 2.6 | 1 | 2.5 | 1.5 | 2.25 | 0.75 | 0.5 | 1.5 | 0.8 | 180 |
| C4 | 0.25 | 0.5 | 2.8 | 2 | 3 | 3.5 | 1.75 | 0.5 | 2.5 | 1.75 | 140 |
| C5 | 0.75 | 0.5 | 2.5 | 1.1 | 1.4 | 1.8 | 1.75 | 0.5 | 2.5 | 3 | 210 |
| C6 | 1.5 | 2.75 | 1.4 | 2 | 3.2 | 3.5 | 1.9 | 0.5 | 2.25 | 0.9 | 190 |
| C7 | 1.4 | 0.7 | 3.2 | 2.3 | 2.1 | 0.6 | 1.6 | 0.3 | 2.9 | 3.1 | 160 |
| C8 | 1 | 1.3 | 3.3 | 3.6 | 0.65 | 1.35 | 1.75 | 0.85 | 2 | 1.95 | 100 |
| C9 | 0.4 | 2.1 | 0.35 | 2.8 | 3.45 | 3 | 1.25 | 0.55 | 2.7 | 0.2 | 205 |
| C10 | 2 | 1.2 | 3.4 | 1.65 | 3.2 | 2.4 | 2.8 | 1.3 | 0.9 | 2.4 | 170 |
| Estimated # requests | 150 | 90 | 80 | 110 | 135 | 95 | 105 | 120 | 75 | 160 |  |

Once the standard formulation has been implemented in Pyomo, the results obtained for our data are:

Objective:

Objective Value: -1705

x[i,j] for company i and query j

Variable:

x[2,5]=7.52688172043 ≈ 8.

x[2,10]=69.3548387097 ≈ 69.

x[3,2]=69.2307692308 ≈ 69.

x[4,6]=40.

x[5,10]=70.

x[6,6]=54.2857142857 ≈ 54.

x[1,5]=65.8333333333 ≈ 66.

x[7,3]=30.

x[1,6]=0.714285714286 ≈ 1.

x[7,10]=20.6451612903 ≈ 21.

x[8,4]=27.7777777778 ≈ 28.

x[9,5]=59.4202898551 ≈ 59.

x[10,3]=50.

The obtained results were given by Pyomo as real numbers, but we wanted also to have the dual variables associated to the constraints of the primal problem, so the numbers were kept with their continuous values, rounding them to the nearest integers, as the number of ads has to be a natural number in real life.

The value of the dual variables associated with the constraints was -1 for all of them. Thus, for every query, if the estimated number of requests per query is incremented in 1 unit, the revenue will not be modified. However, if the budget of any of the companies increases 1€, the total revenue function will increase also in 1€.

# 5. The dual problem

To formulate the dual problem in the general case, for the shake of clarity of the results, we will distinguish between the variables Y that correspond to the companies and those that correspond to the queries. Then, we will use Y1i (i = 1,2, ... n) for the variables of the dual problem associated to the companies, and Y2j (j = 1,2, ... m) for the variables of the dual problem associated to the queries. Therefore, the vector Y can be represented as YT=(Y11, .....,Y1n,Y21,.......Y2m).

The, the objective function will be:

The restrictions will be given by . The structure that AT has will be the following:

AT=

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| C11 | 0 | 0 | 0 | 1 | 0 | . | 0 |
| C12 | 0 | 0 | 0 | 0 | 1 | . | 0 |
| .  . | .  . | .  . | .  . | .  . | 0  . | .  . | .  0 |
| C1,m | 0 | 0 | 0 | 0 | 0 | . | 1 |
| . | C­21 |  | 0 | 1 | 0 |  | 0 |
| .  .  . | .  .  . | .  .  . | .  .  . | 0  .  . | 1  0  . | .  . | 0  .  . |
| 0 | C2,m | . | 0 | 0 | 0 | . | 1 |
| .  . | .  . | .  . | .  . | .  . | .  . | .  . | .  . |
| 0  .  .  0 | 0  .  .  0 | .  .  .  . | Cn,1  .  .  Cn,m | 1  0  .  0 | 0  1  .  . | .  .  .  . | .  .  .  1 |
| 1  0  .  0 | 0  1  .  0 | .  .  .  . | 0  0  .  0 | 0  0  .  0 | 0  0  .  0 | .  .  .  . | 0  0  .  0 |
| .  .  0  0 | .  .  0  0 | .  .  . | .  .  0  1 | .  .  0  0 | .  .  0  0 | .  .  .  . | .  .  0  0 |
| 0  0  .  .  0 | 0  0  .  .  0 | .  .  .  .  . | 0  0  .  .  0 | 1  0  .  .  0 | 0  1  .  .  0 | .  .  .  .  . | 0  0  .  0  1 |

In the end the conditions these are the conditions that we obtain:

C11Y11+Y21<=-C11

C12Y11+Y22<=-C12

....................

C1,nY11+Y2m<=-C1,m

...................

Cn,1Y1n+Y21<=-Cn,1

..................

Cn,mY1n+Y2m<=-Cn,m

Y1i<=0 ( i=1,2,....,n) Y2j<=0 (j=1,2,....,m)

1. **Implement the dual model derived in 4) in Pyomo and solve it for the same data in 3). Report the results**

Once the dual problem has been implemented in Pyomo, the results obtained for our data are:

Objective:

Objective Value: -1705

Variable:

y[10]: Value: -1

y[1]: Value: -1

y[2]: Value: -1

y[3]: Value: -1

y[4]: Value: -1

y[5]: Value: -1

y[6]: Value: -1

y[7]: Value: -1

y[8]: Value: -1

y[9]: Value: -1

Constraint:

cons[20]: Dual: 75

cons[22]: Dual: 69.2307692308

cons[36]: Dual: 40

cons[50]: Dual: 70

cons[55]: Dual: 59.375

cons[5]: Dual: 2.5

cons[63]: Dual: 35.46875

cons[6]: Dual: 55

cons[70]: Dual: 15

cons[74]: Dual: 27.7777777778

cons[85]: Dual: 59.4202898551

cons[93]: Dual: 44.53125

cons[95]: Dual: 5.810546875

# 6. The strong duality theorem

The strong duality theorem states that: “If x\* is the optimal solution of the primal minimization LP problem, and y\* is the dual optimal solution of the corresponding dual maximization LP problem then Z\*D 0 bty\*0ctx\*=z\*p

Comparing the solutions in 3) and 5), check if the Strong Duality Theorem holds. What is the relationship between the sensitivities computed in 3) and the optimal value of the dual variables obtained in 5)?

The strong duality theorem holds because we get the same results for the optimal solution for the primal and the dual problems. The sensitivities obtained in 3) and the variables obtained in the dual problem 5) are the same, as we can see from the results for both questions where we obtained the same values of – 1.

# 7. The modified problem

This might not make any sense as I don’t understand that google now are able to have N ads displayed simultaneously instead of just one. Because x is only restricted by the number of estimated requests? If they were only able to display one ad, should we not have a constraint saying:

For all queries m, the sum of ad for all companies have to be equal to 1.

And then saying that this now can be modified to:

??? What is it that I don’t understand? ☹

Sets

N – set of ads displayed simultaneously

Variables

Including a new binary variable

Constraints

Only one ad for each query j can have a certain order number o

The ad with the highest average revenue will have the highest order (this ordering can also be based on other things)

If an ad have an order between 1 and N, the number of ads have to be greater than or equal to 1.

Aldos’ way:

n = # of companies (i=1,..,n)

m=# of queries (j=1,…,m)

l= # of ads (k=1,…,l)

dj🡪estimated number of request of query j

bi🡪budget of company i

St.